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Some developments in the theory of vortex breakdown

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The primary aim of the analysis presented herein is to consolidate the ideas of the 'conjugate-flow' theory, which proposes that vortex breakdown is fundamentally a transition from a uniform state of swirling flow to one featuring stationary waves of finite amplitude. The original flow is assumed to be supercritical (i.e. incapable of bearing *infinitesimal* stationary waves), and the mechanism of the transition is explained on the basis of physical principles that are well established in relation to the analogous supercritical-flow phenomenon of the hydraulic jump or bore. In previous presentations of the theory the existence of appropriately descriptive solutions to the full equations of motion has only been inferred from these general principles, but here the solutions are demonstrated explicitly by means of a perturbation analysis. This has basically much in common with the classical theory of solitary and cnoidal waves, which is known to explain well the essential properties of weak bores.

In §2 the basic equations of the problem are set out and the leading results of the original theoretical treatment are recalled. The new developments are mainly presented in §3, where an analysis of finite-amplitude waves is completed by two different methods, each serving to illustrate points of interest. The effects of small energy losses and of small flow-force reductions (i.e. wave-resistance effects) are considered, and the analysis leads to a general classification of possible phenomena accompanying such changes of integral properties in either slightly supercritical or slightly subcritical vortex flows. The application to vortex breakdown remains the focus of attention, however, and §3 includes a careful appraisal of some experimental observations on the phenomenon. In §4 a summary is given of a variant on the previous methods which is required when the radial boundary of the flow is taken to infinity. The main analysis is developed without restriction to particular flow models, but in §5 the results are applied to a specific example.

1. Introduction

The theory to which the present work adds has been presented in two previous papers (Benjamin 1962, 1965; hereafter the first of these papers will be referred to as I). According to it, the vortex-breakdown phenomenon is explained as a transition between two steady states of axisymmetric swirling flow, being much the same in principle as the hydraulic jump in open-channel flow. The state upstream from the breakdown point, say flow A, is assumed to have cylindrical stream-surfaces (i.e. to be axially uniform) and to be *supercritical,* which means that stationary waves of infinitesimal amplitude cannot be formed upon it by any non-dissipative process. From this assumption, two important steps in the original analysis were to prove the following properties of the 'conjugate' cylindrical flow B that is generally derivable from A by applying conditions of energy and angular-momentum conservation between the two and also requiring them to satisfy the same boundary conditions. First, B is always *subcritical*, so that stationary periodic waves can arise upon it. Secondly, B in comparison with A has the greater value of flow force S, defined for each as the sum of the total axial momentum-flux and axial pressure force. Hence, the argument proceeds, a flow formed by the superposition of stationary waves on B may exist downstream from the breakdown point, the flow-force balance needed for a steady state being achieved by the effect of wave resistance—which is equivalent to a reduction in flow force. In other words, the excess $S_B - S_A$ is absorbed by wave formation; or alternatively, if the original flow is a long way supercritical and consequently $S_B - S_A$ is large, such a violent wave-making action is induced that the leading wave breaks in the form of a burst of turbulence.

At the time when this argument was first advanced, no theory of finite-amplitude waves in swirling flows was available, and the property of flow-force reduction from a uniform subcritical state could only be inferred from the theory of infinitesimal waves (I, §3.4). Though physically plausible and supported by analogy with well-established results in gravity-wave theory, this line of reasoning therefore lacked a complete mathematical demonstration. It was also somewhat oblique in that the physical realizability of the proposed type of flow following breakdown was deduced in the two steps $A \rightarrow B \rightarrow B + waves$, rather than in the single step $A \rightarrow A + waves$ which might have made a more convincing explanation if some analytical basis for it could have been provided. The main object of the present paper is to demonstrate the latter, more direct kind of argument for the existence of wavy flows arising from vortex breakdown. That is, it will be shown how such flows can be represented as perturbations of finite amplitude from the original supercritical flow A. The required theory has the same place in this context as cnoidal-wave theory in the study of open-channel flows, the classical version of which theory was shown by Benjamin & Lighthill (1954) to account satisfactorily for weak, undular hydraulic jumps.

A suitable method of analysis has been developed in a recent paper (Benjamin 1966; hereafter this paper will be referred to as II), but there it was applied to the description of finite-amplitude waves in non-homogeneous fluids under gravity. A comprehensive account of steady long-wave phenomena in that type of system was attempted, including 'internal' bores and lee waves, and several points with direct bearing on the vortex-breakdown theory were noted. Although the essential ideas have thus already been covered, it is considered worth while to redevelop some of the analysis in the context of swirling flows and so make good the claims for this application that were outlined in II.

2. Definitions and basic equations

A frictionless and incompressible fluid, with density ρ, is considered to flow steadily along a uniform duct whose cross-section is circular and has radius R. Let x and r denote axial and radial co-ordinates, with x increasing in the direction of flow, and let $y = \frac{1}{2}r^2$, $a = \frac{1}{2}R^2$. For the primary, cylindrical state of flow from which waves are supposed to arise, the axial velocity W and swirl velocity V are prescribed functions of y alone. The stream-function $\Psi_A(y)$ for this primary flow is defined by $W = d\Psi_A/dy$, $\Psi_A(0) = 0$, and the pressure p can be found by integrating the equation of radial equilibrium $dp/dy = \frac{1}{2}\rho V^2/y$. Hence the stagnation pressure $\rho H = p + \frac{1}{2}\rho(W^2 + V^2)$ and the quantity $I = yV^2$, which is $(8\pi^2)^{-1}$ times the square of the circulation $2\pi rV$, can be expressed as functions of y or of Ψ_A.

In a second state of axisymmetric flow arising from the primary one without energy loss, H and I keep their original values along the common stream-surfaces. It was shown in I (appendix, §(a)) that these properties require the stream-function $\psi(x, y)$ to satisfy the equation

$$\psi_{yy} + \frac{1}{2y}\psi_{xx} = H'(\psi) - \frac{1}{2y}I'(\psi), \tag{1}$$

which is generally non-linear. Here the accents denote derivatives with respect to ψ, and the functions $H(\psi)$ and $I(\psi)$ have the same forms as $H(\psi_A)$ and $I(\psi_A)$ derived for the primary flow. The kinematical boundary conditions, at the axis and at the wall of the duct, are

$$\psi(x, 0) = 0, \quad \psi(x, a) = \Psi_A(a). \tag{2}$$

In terms of ψ, the radial, azimuthal and axial velocity components are given respectively by

$$u = \frac{1}{(2y)^{\frac{1}{2}}}\psi_x, \quad v = \left\{\frac{I(\psi)}{y}\right\}^{\frac{1}{2}}, \quad w = \psi_y. \tag{3}$$

In I particular attention was paid to the relationships between states of *cylindrical* flow possible in the same system. A solution $\psi = \Psi(y) + \Psi_A(y)$ of the ordinary differential equation

$$\psi_{yy} = H'(\psi) - \frac{1}{2y}I'(\psi) \tag{4}$$

subject to the boundary conditions (2) is said to be *conjugate* to the primary solution Ψ_A. An indefinitely large number of such solutions generally exists, but the vortex-breakdown theory is concerned only with the conjugate state B that is *adjacent* to A (I, pp. 605, 608), being defined uniquely by the property that the solution curves in the (y, ψ)-plane, as illustrated by figure 1, do not intersect except at their end-points $[0, 0]$ and $[a, \Psi_A(a)]$.† An extension of the theory to account for flows in ducts with gradually-varying circular cross-section has been outlined by Benjamin (1965). In this application ψ is still a solution of (4) rather than (1), but depends on x through the second of the boundary conditions (2), in which a is allowed to be a slowly-varying function of x.

Let us now consider *small* axisymmetric perturbations from a given cylindrical flow, assuming them to be imposed under the conditions that the total flow rate (which equals $2\pi\Psi(a)$) and the distributions of H and I with ψ remain unchanged. (The cylindrical flow in question may be either A or B, and so for the moment no suffix will be used to distinguish the particular Ψ.) Supposing a disturbance with

† Another definition of the adjacent conjugate state was also introduced in I (p. 606), and was shown to imply the present one. The equivalence of the two has been proved rigorously by Fraenkel (1966).

exponential dependence on x (including the case of a sinusoidal wave, for which the exponent is purely imaginary), we may put

$$\psi = \Psi(y) + \epsilon\phi(y)e^{\gamma x}, \tag{5}$$

and from (1) derive a linearized equation for ϕ (cf. I, §3.1 and appendix, §§(*b*), (*c*)). This is

$$\frac{d^2\phi}{dy^2} + \left\{\frac{\gamma^2}{2y} + P(y)\right\}\phi = 0, \tag{6}$$

where

$$P(y) = -H''(\Psi) + \frac{I''(\Psi)}{2y} \tag{7}$$

$$= -\frac{W_{yy}}{W} + \frac{I_y}{2y^2W^2}. \tag{8}$$

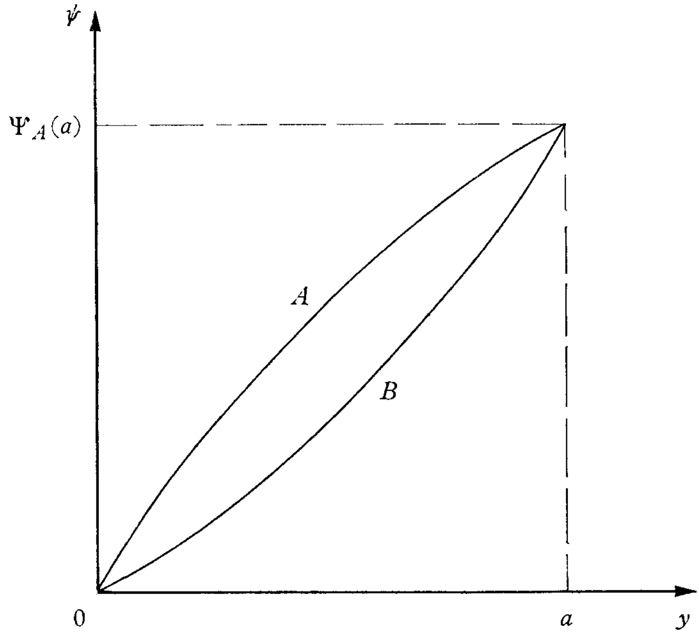


FIGURE 1. Illustration of conjugate solutions $\psi = \Psi_A(y)$ and $\psi = \Psi_B(y)$.

The boundary conditions require that

$$\phi(0) = 0, \quad \phi(a) = 0. \tag{9}$$

A helpful classification of flow properties can be made by consideration of (6) and (9) as a Sturm–Liouville system (see I, §3.1; also II, §3.4). A feature calling for caution is that the system is singular at the end-point $y = 0$, where both P and the coefficient of the eigenvalue γ^2 have simple poles (P because the angular velocity ω at the axis will be finite in any practical steady flow with swirl, so that $I \to 2\omega^2y^2$ for $y \to 0$). However, it can be confirmed that the required conclusions from Sturm–Liouville theory (in particular, the existence of an infinite sequence of real eigenvalues $\gamma_0^2 < \gamma_1^2 < \gamma_2^2 < \ldots$) are unaffected by this feature: for instance, an adequate foundation for them may be found in the simple exposition of the theory by Morse & Feshbach (1953, p. 719 *et seq.*), and for a rigorous discussion of the 'singular case' reference may be made to the treatise by Titchmarsh (1962). If at least one eigenvalue, say γ_0^2, is negative, then the cylindrical flow represented by $\Psi(y)$ is said to be *subcritical*. A subcritical flow can therefore support, under the assumed conditions, an infinitesimal stationary disturbance in the form of a sinusoidal wave whose wavelength is given by $2\pi/\alpha_0$, where $\alpha_0^2 = -\gamma_0^2$. If every eigenvalue γ^2 is positive, the flow is said to be *supercritical*. Thus, under the assumed conditions, stationary sinusoidal waves are impossible on a supercritical flow.† It was shown in I that if flow A is supercritical, then the conjugate flow B is subcritical, and this universal property of conjugate-flow pairs, if they exist, has been reaffirmed by Fraenkel (1966) using a different argument.

In this paper we are concerned only with flow conditions close to critical, so that the magnitude of the least eigenvalue γ_0^2 may be regarded as small. Recalling how γ was introduced in (5), we may introduce a typical length scale l for the axial variations of a small perturbation and suppose l to be much larger than the radius R of the duct. Then we express the eigenvalue by

$$\gamma_0^2 = l^{-2}\Gamma^2, \tag{10}$$

and regard Γ as $O(1)$ when the radius of the duct is taken as the unit of length. As a well-known general property of Sturm–Liouville functions, the respective solution ϕ_0 of (6) and (9) makes only one oscillation over the interval, thus having no zero between the end-points.

It can also be supposed that the condition of any flow in question could be made critical by slightly changing the value of some physical parameter. This parameter might be chosen in several ways and at present there is no need to be specific about it, but let us denote it by c. Then the following considerations show that the fractional change, say $\delta c/c$, needed to make the flow critical is $O(l^{-2})$. We write

$$\begin{aligned}\tilde{P}(y) - P(y) &= \frac{\partial P}{\partial c}\delta c + O(l^{-4}) \\ &= l^{-2}\zeta(y) + O(l^{-4}),\end{aligned} \tag{11}$$

where $\tilde{P}(y)$ is defined as the modified form of $P(y)$ representing a critical flow; thus there exists a solution of

$$(d^2\tilde{\phi}/dy^2) + \tilde{P}\tilde{\phi} = 0 \tag{12}$$

† These ideas are illustrated by the following example. Let $W = C$ (const.) and $I = 2\omega^2 y^2$ throughout the closed interval between $y = 0$ and $y = a$. Then equation (6) becomes

$$\phi_{yy} + \tfrac{1}{4}y^{-1}\beta^2\phi = 0, \quad \text{with} \quad \beta^2 = (4\omega^2/C^2) + 2\gamma^2.$$

The solution vanishing at $y = 0$ is

$$\phi = (2y)^{\frac{1}{2}} J_1\{\beta(2y)^{\frac{1}{2}}\} \equiv rJ_1(\beta\gamma).$$

Hence the other end-condition $\phi = 0$ at $r = R$ leads to

$$\gamma_n^2 = \tfrac{1}{2}(\gamma_n^2/R^2) - (2\omega^2/C^2) \quad (n = 0, 1, 2, \ldots),$$

where the λ_n are the successive positive zeros of the Bessel function J_1. Thus, since $\lambda_0 = 3{\cdot}832$, the flow is subcritical if $\omega R > 1{\cdot}916C$ and supercritical if $\omega R < 1{\cdot}916C$.

Note that the second solution of the differential equation is $rY_1(\beta r)$, which is unbounded at $r = 0$; but this solution is excluded as part of the eigenfunction ϕ by the first end-condition. As exemplified here, equation (6) in general has independent solutions in the forms $y\mathbf{A}$ and $y(\mathbf{B} + \mathbf{A}\log y)$, where $\mathbf{A}$ and $\mathbf{B}$ are ascending power series each of which starts with a constant term; but the second of these solutions is excluded.

satisfying

$$\left.\begin{aligned}\tilde{\phi} &= 0 \quad \text{for} \quad y = 0, a, \\ &\neq 0 \quad \text{for} \quad 0 < y < a.\end{aligned}\right\} \tag{13}$$

To relate the physical change $\zeta(y)$ to the value of the eigenvalue γ_0^2, we now express the original eigenfunction by the expansion

$$\phi_0 = \xi_I + l^{-2}\xi_{II} + l^{-4}\xi_{III} + \dots. \tag{14}$$

Substituting this expansion, together with (10) for γ_0^2 and (11) for P, in (6) and (9) and then separating the coefficients of successive powers of l^{-2}, we first find that

$$\xi_I = \tilde{\phi}. \tag{15}$$

Then we obtain

$$\left.\begin{aligned}\frac{d^2\xi_{II}}{dy^2} + \tilde{P}\xi_{II} &= \left(\zeta - \frac{\Gamma^2}{2y}\right)\xi_I, \\ \xi_{II}(0) = \xi_{II}(a) &= 0.\end{aligned}\right\} \tag{16}$$

A necessary and sufficient condition for the inhomogeneous system (16) to have a solution is that the right-hand side of the differential equation should be orthogonal to the solution $\tilde{\phi}$ of the homogeneous system (Ince 1926, §1.32). Thus, using (15), we conclude that

$$\int_0^a \zeta\tilde{\phi}^2\,dy = \tfrac{1}{2}\Gamma^2 \int_0^a \frac{\tilde{\phi}^2}{y}\,dy. \tag{17}$$

This formula confirms the previous assumption about the magnitude of $\tilde{P} - P$, and it can be used to determine the scaled eigenvalue Γ^2 from the physical property ζ.†

Finally, a definition of the *flow force* is needed:

$$S = 2\pi \int_0^a (p + \rho w^2)\,dy. \tag{18}$$

Using the fact that

$$H(\psi) = (p/\rho) + \tfrac{1}{2}(u^2 + v^2 + w^2),$$

and substituting the expressions (3) for the velocity components, we obtain from (18)

$$S = 2\pi\rho \int_0^a \left\{\tfrac{1}{2}\psi_y^2 - \frac{\psi_x^2}{4y} + H(\psi) - \frac{I(\psi)}{2y}\right\} dy. \tag{19}$$

By differentiation of this integral with respect to x, it appears directly that (1) implies $S = \text{const.}$ if $a = \text{const.}$ (cf. II, §2.2). This confirms the obvious physical property that S is an invariant of any frictionless steady flow along a uniform duct.

† Note that for the example explained in the preceding footnote, we have

$$l^{-2}\zeta = \frac{1}{y}\left\{\frac{(1\cdot916)^2}{R^2} - \frac{\omega^2}{C^2}\right\},$$

and evidently the 'critical' form $\tilde{P}$ of P, as represented by the first term on the right-hand side, can be realized either by increasing ω or decreasing C from respective supercritical values. In this case, (17) gives

$$\gamma_0^2 = 2R^{-2}[(1\cdot916)^2 - (\omega R/C)^2],$$

reproducing the formula given in the first footnote.

By means of a variational argument applied to the S-integral for cylindrical flows (i.e. (19) without the second term in the integrand), it was shown in I (§§4.3, 4.5) that $S_B > S_A$ always, if A and B are respectively the supercritical and the subcritical member of a conjugate pair of flow states. This conclusion has been established in another way by Fraenkel (1966).

3. Analysis for long waves of finite amplitude

The procedure to be followed is the same in principle as the one that was developed in II, and reference may be made to the previous paper for supporting discussion. We put

$$\psi = \Psi_A(y) + \epsilon\hat{\psi}(x, y; \epsilon), \tag{20}$$

obtaining from (2) the boundary conditions

$$\hat{\psi} = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad y = a. \tag{21}$$

We then expand the flow-force integral (19) in powers of ϵ as far as ϵ^3, at which stage the implicit dependence of $\hat{\psi}$ on ϵ is decided. Expressed by this expansion, the condition that S is a specific constant determines the approximate form of a stationary-wave disturbance for small but finite values of the amplitude parameter ϵ. In the appendix to this paper, it will be shown how the results obtained in this way accord with the results given by a more orthodox procedure of expansion in powers of ϵ, wherewith the differential equation (1) for the stream-function is solved by successive approximations. The present approach is more fruitful than this alternative, however, since it reveals the precise physical significance of a certain parameter which in the other approach appears merely as an arbitrary constant of integration.

Allowance for the effect of a possible axial force applied externally, as when a rigid body is fixed in the flow or when there is a frictional force, can be made by including a 'wave-resistance' term in the flow-force balance; but this is assumed to be only a small quantity $O(\epsilon^3)$. Accordingly we write

$$S_A - S = 2\pi\rho\epsilon^3 s \tag{22}$$

(cf. II, §3.5, assumption (iii)). A correspondingly small loss of energy, causing a diminution in the total head H, can also be represented consistently within the framework of the third-order approximation. We therefore write

$$\int_0^a (H_A - H)\, dy = \epsilon^3 q \tag{23}$$

(cf. II, §3.5, assumption (iv)). [It should be recognized that dissipative processes might also cause some change in the distribution of angular momentum, i.e. in the form of I; but this effect could also be included by an obvious extension of the definition of q. For a full explanation of the principle in view, see the footnote on p. 253 of II.]

Using these definitions and expanding the integral expression (19) after substitution of (20), we find that the coefficient of ϵ^1 in $S-S_A$ vanishes identically (see I, appendix, §(c)) and we obtain to $O(\epsilon^3)$

$$\frac{S-S_A}{2\pi\rho} = -\epsilon^3 s = -\epsilon^3 q + \tfrac{1}{2}\epsilon^2 \int_0^a \left\{\hat{\psi}_y^2 - \frac{\hat{\psi}_x^2}{2y} - K''(\Psi_A, y)\hat{\psi}^2\right\} dy + \tfrac{1}{6}\epsilon^3 \int_0^a K'''(\Psi_A, y)\hat{\psi}^3 dy, \tag{24}$$

where
$$K''(\Psi_A, y) \equiv \left[\frac{\partial^2}{\partial z^2}\left\{-H(z) + \frac{I(z)}{2y}\right\}\right]_{z=\Psi_A(y)},$$
$$K'''(\Psi_A, y) \equiv \left[\frac{\partial^3}{\partial z^3}\left\{-H(z) + \frac{I(z)}{2y}\right\}\right]_{z=\Psi_A(y)}.$$

This may be treated as an integral equation for $\hat{\psi}$, in which q and s are regarded as prescribed parameters. To solve the equation approximately and apply it to the description of vortex breakdown, we may proceed in two alternative ways as follows.

Method 1

Put
$$\hat{\psi} = f(X)\chi(y), \tag{25}$$
with
$$X = \epsilon^{\frac{1}{2}} x, \tag{26}$$
and
$$\epsilon\chi = \Psi_B - \Psi_A. \tag{27}$$

Here χ is understood to be of the same order of magnitude as Ψ_A and Ψ_B, and (26) is equivalent to the assumption that $l = \epsilon^{-\frac{1}{2}}$. The possible validity of this form of solution, and of the underlying assumption that ϵ is a small number, clearly requires the state of flow A to be close to critical, because only then is the conjugate flow B little different from A. Note that the transformation (26) to a stretched axial scale with $l = \epsilon^{-\frac{1}{2}}$ corresponds to a well-known principle in classical solitary- and cnoidal-wave theory (cf. II, §1 and §3.5, assumption (i)).

By definition both Ψ_A and $\Psi_B = \Psi_A + \epsilon\chi$ are solutions of (4) satisfying the boundary conditions (2). Hence, by expanding in powers of ϵ, we find from (4) that

$$\chi_{yy} = -K''(\Psi_A, y)\chi - \tfrac{1}{2}\epsilon K'''(\Psi_A, y)\chi^2 + O(\epsilon^2),$$

and so, since $\chi(0) = \chi(a) = 0$,

$$\int_0^a \{\chi_y^2 - K''(\Psi_A, y)\chi^2\} dy = \tfrac{1}{2}\epsilon \int_0^a K'''(\Psi_A, y)\chi^3 dy + O(\epsilon^2). \tag{28}$$

This result is now used to reduce equation (24) after substitution of (25). Thus all terms of the equation are seen to be $O(\epsilon^3)$, and upon rearrangement it gives

$$C f_X^2 = D f^2(\tfrac{3}{2} - f) + 2(s-q), \tag{29}$$

in which
$$C = \frac{1}{2}\int_0^a \frac{\chi^2}{y} dy, \tag{30}$$
$$D = \frac{1}{3}\int_0^a K'''(\Psi_A, y)\chi^3 dy. \tag{31}$$

The subsequent physical interpretation depends crucially on the fact that the coefficient D is always positive when flow A is supercritical, the proof of which is as follows. By putting $\psi = \Psi_B = \Psi_A + \epsilon\chi$ in the flow-force integral (19), expanding powers of ϵ and reducing the result by use of (28) in the same way as before, we readily establish that

$$S_B - S_A = \tfrac{1}{2}\pi\rho\epsilon^3 D + O(\epsilon^4). \tag{32}$$

We therefore have $D > 0$ as a corollary of the general theorem that $S_B > S_A$, which was recalled at the end of §2.

The result (29) is a sufficient basis for the description of the wavy flows envisaged to arise from a mild vortex breakdown, but a detailed explanation is postponed until an alternative equation of this type has been derived by method 2 below. We note at once, however, that with $s = q = 0$ the only non-trivial solution of (29) is

$$f = \tfrac{3}{2}\,\mathrm{sech}^2\{\tfrac{1}{2}(\tfrac{3}{2}D/C)^{\frac{1}{2}}X\}, \tag{33}$$

which represents a solitary wave. Thus it appears that a solitary wave is the only steady disturbance that can arise in a supercritical swirling flow without change of energy or flow force. The same property has previously been established for supercritical flows along horizontal open channels (Benjamin & Lighthill 1954) and for supercritical flows of arbitrarily stratified fluids (II, §3.6). To account for the formation of steady periodic waves upon a supercritical swirling flow, it is necessary to assume some slight loss of energy, just as was shown in the previous studies to be necessary to account for undular hydraulic jumps and for internal bores.

A point of interest is that, according to (25) and (33), the amplitude of the solitary wave possible in a given slightly supercritical flow is $\frac{3}{2}$ times the perturbation giving the conjugate subcritical flow. Thus, in the wave, the changes in cross-sectional structure swing 50 % beyond the conjugate state. In so far as we are at present concerned only with a first approximation to long waves of finite amplitude and permanent form, this property seems to be universal among the general class of physical systems in which solitary waves are possible. For instance, it is easily shown that the maximum elevation of a solitary wave on water in an open channel is $\frac{3}{2}$ times the rise in water level to the respective conjugate state (i.e. to the uniform subcritical flow that has the same discharge and total head, but not flow force, as the supercritical flow in question).

Method 2

In applications to specific examples, a disadvantage of the preceding results is that the non-linear ordinary differential equation (4) needs to be solved in order to find χ and hence determine the coefficients C and D. The present approximate method requires only a linear equation to be solved.

We try for a solution of (24) in the form

$$\hat{\psi} = g(X)\tilde{\phi}(y), \tag{34}$$

where $\tilde{\phi}$ is defined by (12) and (13). To reduce the result obtained after substitution of this expression into (24), use is made of the assumption that the condition of flow A is close to critical, the margin being $O(\epsilon)$. Thus, putting $l^{-2} = \epsilon$ in (11), we write

$$P(y) = \tilde{P}(y) + \epsilon\zeta(y), \tag{35}$$

and note from (7) that $P(y)$ is the same as $K''(\Psi_A, y)$. Also for use in the reduction of (24), we obtain from (12) and (13)

$$\int_0^a \left\{ \left(\frac{d\tilde{\phi}}{dy}\right)^2 - \tilde{P}(y)\tilde{\phi}^2 \right\} dy = 0. \tag{36}$$

Hence (24) is seen to be satisfied by (34) to $O(\epsilon^3)$ if

$$Eg_X^2 = Fg^2 - Gg^3 + 2(s-q), \tag{37}$$

where

$$E = \frac{1}{2}\int_0^a \frac{\tilde{\phi}^2}{y}\, dy, \tag{38}$$

$$F = \int_0^a \zeta\tilde{\phi}^2 dy = \Gamma^2 E, \tag{39}$$

$$G = \frac{1}{3}\int_0^a K'''(\Psi_A, y)\tilde{\phi}^3 dy. \tag{40}$$

Note that the second of (39) follows from (17), and that the arbitrary constant multiplying $\tilde{\phi}$ as defined does not affect the result obtained from (34) and (37). Equation (37) is of the same type as (29) which was derived by method 1, and the physical interpretation that will now be made with regard to (37) can easily be refashioned on the basis of the previous equation. The respective end-results are, of course, equivalent within the adopted order of approximation (e.g. apart from a constant factor, the functions $\tilde{\phi}$ and χ are approximately the same), and the only essential difference is one of computational advantage. A helpful feature of the present equation is that the effect of the extent to which flow A departs from a critical condition is represented explicitly in the coefficient F defined by (39), whereas before it was merely implicit in the function χ.

We have $F > 0$ whenever the primary flow is supercritical, since then $\Gamma^2 > 0$ as was explained in § 2. This is the case principally in view. But we also observe that $F < 0$ when the primary flow is subcritical (for which case the identifications A and B need to be reversed between the conjugate-flow pair engirdling the critical state), and the interpretation of (37) in this case seems worth noting incidentally. The two cases require separate discussion as follows, the argument being essentially the same as in II, § 3.7. To be definite we assume that $G > 0$, but the general conclusions to be drawn are obviously unchanged when $G < 0$.

Supercritical case

Since $F > 0$, the cubic in g on the right-hand side of (37) has the form illustrated in figure 2(*a*). When $q - s = 0$, curve $\mathscr{A}$ is described, touching the g-axis at the origin. The solution of (37) is then

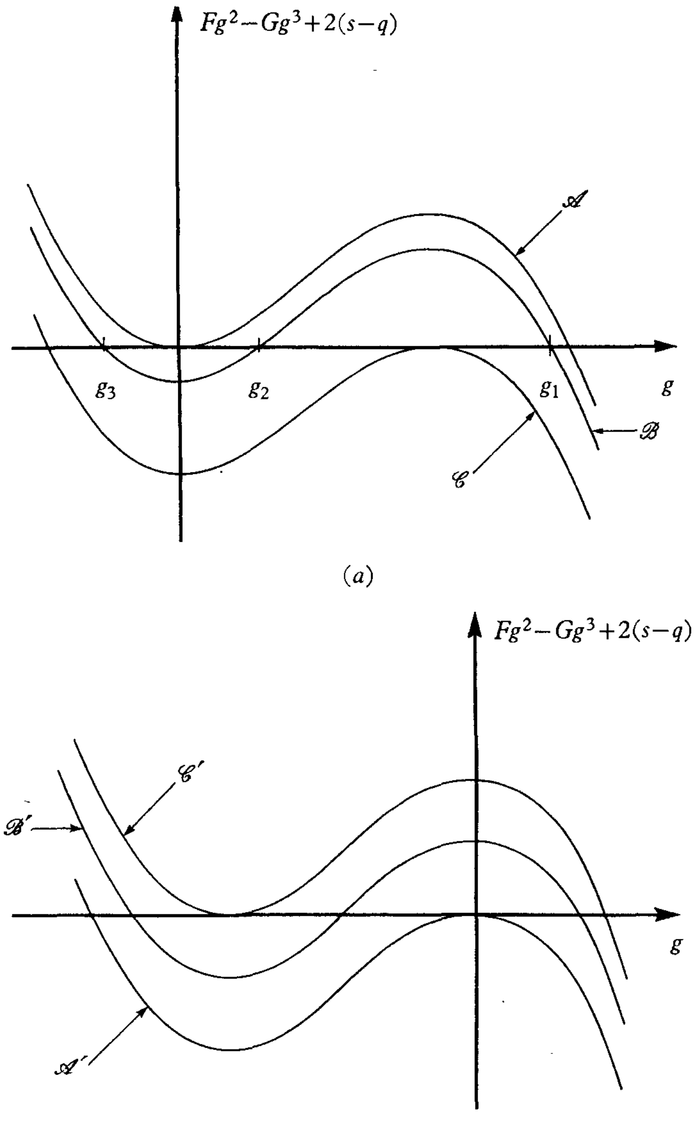
$$g = \frac{F}{G}\operatorname{sech}^2\left\{\frac{1}{2}\left(\frac{F}{E}\right)^{\frac{1}{2}} X\right\}, \tag{41}$$

which like (33) represents a solitary wave. By comparing the expression for $\hat{\psi}$ given by (25) and (33) with that given by (34) and (41), we deduce that a first approximation to the function χ is

$$\chi(y) = (2F/3G)\phi_0 + O(\epsilon), \tag{42}$$

which is independent of the arbitrary constant multiplying ϕ_0.

It seems reasonable to suppose that the disturbance heading a mild vortex breakdown may closely resemble a solitary wave, just as does the leading wave of a weak, undular hydraulic jump (Benjamin & Lighthill 1954). The photographic



(b)

FIGURE 2. Forms of the cubic $Fg^2 - Gg^3 + 2(s-q)$ with $G > 0$: (a) supercritical case $F > 0$; (b) subcritical case $F < 0$.

observations made by Harvey (1962, figures 2–4 (plates 1, 2)) on mild breakdowns support this view, since they show the initial disturbance to comprise a nearly symmetrical swelling of the stream-surfaces originally close to the axis, enclosing an ovoid region of circulating fluid. A disturbance thus large enough to produce a stagnation point on the axis (at the front of the ovoid region) is probably beyond the scope of the present approximate theory, whose validity is ensured only when the amplitude ϵ of the flow perturbation is small; but this experimental evidence of a stationary wave-like, axisymmetric disturbance appears wholly in accord with the basic ideas of the theory. Harvey's figure 4 (plate 2) is particularly striking as it shows a bulbous disturbance almost perfectly symmetrical about its equatorial plane. This was produced by making a slight reduction in the cross-section of the duct (i.e. a reduction in a) immediately downstream, the effect of which was to sweep away the similarly bulbous, but much less regular disturbance that otherwise formed behind the first one. Note that, according to the analysis outlined by Benjamin (1965), a contraction of the boundary changes a supercritical flow of this type in the direction away from critical. Hence, as is a well-known effect of such a change in the analogous cases of hydraulic-jump formation and of shock-wave formation in steady gas flows, waves are less liable to be precipitated.

Since in practice the process of vortex breakdown is bound to be dissipative to some extent, it is natural to allow for an energy loss, putting $q > 0$, in applying the theory to the interpretation of the phenomenon. But as the phenomenon evidently may arise without significant external force being present to bring about a change in flow force, we may still take $s = 0$ as the most reasonable specification of the theoretical model, thus ignoring the possible effect of a frictional force (or, perhaps a little more realistically, we might allow for the latter possibility but assume that $q > s$). With $s = 0$, $q > 0$, the curve in figure 2 (*a*) is *lowered* to become one of type $\mathscr{B}$. The cubic then has three distinct real roots and the solution of (37) is periodic: g oscillates between the values g_1 and g_2, and g_X (which by (37) is proportional to either square root of the cubic) changes sign as g passes through these extrema. We thus have an explicit representation of a finite-amplitude wave train arising from the primary supercritical flow, which is precisely the situation propounded by the original presentation of the vortex-breakdown theory in I—although there the necessity of a slight energy loss was not recognized.† (The representation of the energy loss as being concentrated at the front of the wave train is, of course, a convenient idealization which is open to obvious objections; but at least it serves as a rational model for a slightly dissipative system and admits the theory to the description of an important class of physical situations that would be inaccessible if no dissipation were allowed.)

As already mentioned above, Harvey's experiments showed the first wave of a mild vortex breakdown to be followed by a second one unless special measures were taken to suppress it; and there appears reason enough to suppose that a periodic solution of the finite-perturbation equations comes closest, within the limitations of ideal-fluid theory, to representing the observed situation. This interpretation is the same in principle as the one that was made by Benjamin & Lighthill (1954) with regard to undular hydraulic jumps and bores, the applicable theoretical result in that instance being the classical cnoidal-wave solution of Korteweg & de Vries (Lamb 1932, §253). Unlike the analogous phenomenon in

† But see the footnote on p. 597 of I.

open-channel flow, however, the structure of a vortex breakdown in practice manifests only the rudiments of a wave train: the motion becomes unsteady and irregular by the stage of the second wave, if not before, and it appears that the axisymmetric form of disturbance sometimes observed (as by Harvey) is generally unstable and so liable to disintegrate rapidly. Thus the steady wave train that is theoretically possible is never seen intact, although the experimental observations by Harvey and others strongly suggest its inherent role as the basic state upon which the actual flow subsists, developing as a result of instability or of incidentally imposed fluctuations. Nevertheless, such a wave train might be realized much more distinctly as a travelling disturbance, the counterpart of a progressive bore just as the usual vortex-breakdown phenomenon is the counterpart of a stationary hydraulic jump. An experimental investigation into this possibility, and also into the subject of progressive solitary waves in rotating fluids, is currently being undertaken by Mr W. G. Pritchard at Cambridge.

The curve in figure 2(*a*) is of type $\mathscr{B}$ when $0 < q-s < 2F^3/27G^2$. The three roots of the cubic, as indicated in the figure, are given in this case by

$$\left.\begin{aligned} g_1 &= F/3G\{1+2\cos\theta\},\\ g_2 &= F/3G\{1+2\cos(\theta-\tfrac{2}{3}\pi)\},\\ g_3 &= F/3G\{1+2\cos(\theta+\tfrac{2}{3}\pi)\},\\ \text{with}\quad \cos 3\theta &= 1-\frac{27(q-s)G^2}{F^3}\quad (0<\theta<\tfrac{1}{3}\pi). \end{aligned}\right\} \tag{43}$$

The solution of (37) is then

$$\left.\begin{aligned} g &= g_2+(g_1-g_2)\,\mathrm{cn}^2(mX;\,k),\\ \text{with}\quad m^2 &= \frac{G(g_1-g_3)}{4E} = \frac{F}{2\sqrt{3E}}\sin(\theta+\tfrac{1}{3}\pi),\\ k^2 &= \frac{g_1-g_2}{g_1-g_3} = \frac{\sin(\tfrac{1}{3}\pi-\theta)}{\sin(\tfrac{1}{3}\pi+\theta)}. \end{aligned}\right\} \tag{44}$$

The period of the function cn^2 is 2 times $K(k)$, the complete elliptic integral of the first kind, and so the wavelength is

$$\lambda = [2K(k)]/m \tag{45}$$

when expressed in the same units as X.

The solution (44) reduces correctly to the solitary-wave solution (41) when $q-s \to 0$, so that $g_2 \to -g_3 \to 0$ and hence $k \to 1$. It is of interest to examine the form of the solution when $q-s$ is still positive but is very much smaller than $2F^3/27G^2$ (e.g. in the case where $s = 0$ and the energy loss is reduced almost to zero). The solution then resembles a sequence of solitary waves, whose amplitude is very nearly equal to F/G, and the form of the individual waves is scarcely affected by the value of $q-s$. Their spacing λ remains a definite function of $q-s$, however. Using the asymptotic property of $K(k)$ for $k \to 1$ (Whittaker & Watson 1927, p. 521), we deduce that

$$\lim_{q-s\to 0}\left[\lambda-\frac{2E^{\frac{1}{2}}}{F^{\frac{1}{2}}}\log\left\{\frac{2F^3}{(q-s)G^2}\right\}\right] = 0. \tag{46}$$

Thus, although λ is unbounded in the limit $q-s \to 0$, its rate of increase with decreasing $q-s$ is in the end very slow. This result indicates that an exceedingly small amount of dissipation may be enough to account for the periodicity of the flow structure following a mild vortex breakdown, notwithstanding that the second wave could not form within a finite distance if the system were strictly non-dissipative.

Note that the wave amplitude $g_1 - g_2$ diminishes with increasing $q-s$, and vanishes when $q-s = 2F^3/27G^2$. Curve $\mathscr{C}$ is then described in figure 2(*a*), and the double root of the cubic—i.e. $g_1 = g_2 = 2F/3G$—represents the uniform flow that is conjugate to the original flow A (cf. equation (42)). As $q-s$ approaches $2F^3/27G^2$, we have $k \to 0$ and the solution (44) then represents a sinusoidal wave train of infinitesimal amplitude. Thus the conjugate flow is shown to be subcritical, in confirmation of the theorem proved in I. The physical change represented by $\mathscr{A} \to \mathscr{C}$ in figure 2(*a*) corresponds to the classical model for a dissipative hydraulic jump or bore (Lamb 1932, §280), in which a transition from a supercritical uniform flow to a subcritical one is brought about by a total-head loss at constant flow-force. If $s = 0$, the value $r = 2F^3/27G^2$ required to lower the cubic curve from $\mathscr{A}$ to $\mathscr{C}$ represents the maximum energy loss then possible at the front of a vortex breakdown. A large-amplitude wave train can occur (or at least tend to develop in the manner shown by Harvey's observations) only if the energy loss is substantially less than this maximum.

Subcritical case

Since $F < 0$ in this case, the cubic in g has the form illustrated in figure 2(*b*). There is now no non-trivial real solution when $q-s = 0$ (curve $\mathscr{A}'$). This fact means that no steady wave can arise in a subcritical swirling flow without change of energy or flow force, which is a property in common with subcritical open-channel flows (Benjamin & Lighthill 1954) and subcritical flows of stratified fluids (II, §3.7). To obtain waves we must have $s-q > 0$, so that the curve in figure 2(*b*) is *raised* to become one of type $\mathscr{B}'$ and, with obvious modifications, the cnoidal-wave solution (44) applies. This result represents the phenomenon of wave resistance in swirling flows, about which much was said in I. As the simplest physical illustration, suppose an axisymmetric obstacle to be fixed in the subcritical flow and to generate a wave train in its wake. Then the drag on the obstacle equals the flow-force reduction $2\pi\rho\epsilon^3 s$ in the receding flow. Since the wave amplitude (i.e. the spacing of the two higher roots, between which the cubic is positive) obviously increases steadily with $q-s$, we conclude that for a given drag the amplitude is diminished by energy losses as represented by q. Alternatively, we can say that the drag experienced by the obstacle increases both with the amplitude of the waves generated and with incidental dissipation.

For a sufficiently large value of $s-q$, the curves of type $\mathscr{B}'$ in figure 2(*b*) approach curve $\mathscr{C}'$, whose minimum point gives a double root of the cubic and so, like the origin for curve $\mathscr{A}$ in figure 2(*a*), represents the uniform supercritical flow that is conjugate to the subcritical flow in question. As discussed previously, waves of large amplitude and wavelength, resembling a succession of solitary waves, arise under conditions neighbouring on those for a uniform supercritical state as given by $\mathscr{C}'$ or $\mathscr{A}$. When such waves are produced in the way now contemplated, however, it is possible that a very slight change in conditions would result in the wave train being swept away downstream, leaving the flow behind the obstacle in the uniform supercritical state (see Benjamin (1956) for an account of the corresponding phenomenon in an open-channel flow spanned by an obstacle such as a planing plate).

4. The case of unbounded flows

A variant on the preceding methods of treatment will now be outlined, which becomes necessary when the flow is unbounded radially. The preceding methods break down in this case because the assumption that the length scale $l = \epsilon^{-\frac{1}{2}}$ greatly exceeds the radius of the duct is obviously inadmissible, and the alternative assumption that l greatly exceeds some finite radius characteristic of the flow structure (for instance the radius of a core outside which the flow is irrotational—see below) is found to be inadequate.

However, it was shown by an example in I (§5, example 2) that the leading conclusions of the original theory still hold when the boundary is expanded to infinity. The model in question for the primary flow A consists of a core of fluid in solid-body rotation and an outer region of arbitrary extent in which the flow is irrotational, having constant azimuthal circulation. Respective to a supercritical flow A with uniform axial velocity, there was shown to exist a subcritical conjugate flow B in which the core is expanded from its original radius, and various properties of B, such as the flow force excess $S_B - S_A$, appeared to remain determinate in the limit $a \to \infty$. But when the coefficients C defined by (30) and E defined by (38) are formally determined for this model, both are found to increase without bound like $\log a$ as $a \to \infty$. On the other hand, the coefficient D appearing in equation (29) and both F and G in (37) take finite values in this limit, because over the region where the flow is irrotational there is no contribution to the integrals (31), (39) and (40).

One may conclude from these facts that when the flow is extended radially without limit, a solitary wave is still a formally possible solution of the perturbation equations, but its length scale, which equations (29) and (37) show to be proportional to $(C/D)^{\frac{1}{2}}$ or $(E/F)^{\frac{1}{2}}$, becomes indefinitely great. The argument based on methods 1 or 2, proposing a direct connexion between the solitary-wave solution and the flow following a vortex breakdown, is therefore clearly useless in this case. An equally obvious aspect of the difficulty is that there is no choice of the parameter $q - s$ giving a determined periodic solution of (29) or (37).

A way round the difficulty is suggested by the fact, demonstrated in I, that the conjugate flow B remains determinate in the limit $a \to \infty$ and, being subcritical, it can support infinitesimal waves of *finite* wavelength. Accordingly, a perturbation of finite amplitude from flow B is considered, the length scale of which is taken to have the same order of magnitude as the determinate scale of the infinitesimal waves. This approach in effect retraces the original physical argument used in I.

We put
$$\psi = \Psi_B(y) + \epsilon h(X)\phi_{0,B}(y), \tag{47}$$
where $\phi_{0,B}$ is the first eigenfunction of the system (6) and (9) when $P(y)$ is evaluated for flow B. The respective eigenvalue γ_0^2 ($= -\alpha_0^2 = \epsilon\Gamma_B^2$, say) is negative since B is subcritical. Hence, proceeding as before to expand the flow-force integral, we obtain finally
$$L(h_X^2 - \Gamma_B^2 h^2) - Mh^3 = \tfrac{1}{2}D + 2(s-q), \tag{48}$$
in which
$$L = \frac{1}{2}\int_0^a \frac{\phi_{0,B}^2}{y}\,dy, \tag{49}$$
$$M = \frac{1}{3}\int_0^a K'''(\Psi_B, y)\phi_{0,B}^3\,dy, \tag{50}$$
and D is given by (32). It can be easily confirmed that this result is approximately equivalent to the ones given by methods 1 and 2 in cases where the flow has a finite boundary. But the coefficients of (48) all remain determinate when a is made infinite. In this case the outer boundary condition on the solution of (6) must be taken as $\phi_{0,B} \to 0$ for $y \to \infty$; and hence it is found that in the region of irrotational flow $\phi_{0,B} = rK_1(\alpha_0 r)$, where K_1 is the first-order modified Bessel function of the second kind. Unlike C, E or L itself in the case of finite a, however, L is not necessarily independent of ϵ to a first approximation: in fact, for sufficiently small ϵ, it becomes proportional to $-\log(\epsilon\Gamma_B^2)$.

5. An example

This was previously treated in I, §5, where the properties of the flow B conjugate to the given flow A were derived. The solitary-wave solution given by (34) and (41) will now be worked out explicitly, the corresponding results for the cnoidal-wave solution (44) then being obvious. Though it is unrealistic as a model for a stationary vortex breakdown, this model serves well to illustrate the steady propagation (observed in a moving reference frame) of finite waves into a region where the fluid has zero axial velocity and a stable, non-uniform distribution of angular velocity—such as would be created if the containing tube were whirled for a limited time at constant speed.†

For the primary flow A, we take
$$\left.\begin{aligned} W = 1 \quad \text{so that} \quad \Psi_A &= y, \\ \text{and} \qquad V &= (\tfrac{2}{3})^{\frac{1}{2}}\kappa y. \end{aligned}\right\} \tag{51}$$
Hence
$$I = yV^2 = \tfrac{2}{3}\kappa^2 y^3 \equiv \tfrac{2}{3}\kappa^2\Psi_A^3. \tag{52}$$
We also have (cf. I, equation (A 18))
$$H'(\Psi_A) = \frac{I_y}{2yW} \equiv \kappa^2\Psi_A, \tag{53}$$

† It might well be asked why the simple example discussed in the footnotes to §2 is not considered further. This example would be relevant to the experiment just suggested if the tube were whirled for a long time, so that the whole fluid acquired solid-body rotation. But Benjamin & Barnard (1964) have pointed out that, in consequence of equation (1) for the stream-function being *linear* in this case, steady wave propagation into the undisturbed fluid is impossible and there is no solution modelling a steady vortex breakdown (see also II, §3.9, for a discussion of the peculiarities of linear systems).

and deduce from (52) and (53) that

$$K'''(\Psi_A, y) = 2\kappa^2/y. \tag{54}$$

It follows from the preceding definitions that, according to (7) and (8),

$$P(y) = \kappa^2, \tag{55}$$

and clearly

$$\tilde{P}(y) = \tilde{\kappa}^2 = (\pi/a)^2 \tag{56}$$

since this choice of P gives a solution

$$\tilde{\phi} = \sin \tilde{\kappa} y \tag{57}$$

of (12) satisfying the conditions (13). The condition that flow A be supercritical is evidently $\kappa^2 < \tilde{\kappa}^2$.

From (35) we obtain

$$\epsilon\zeta = \tilde{\kappa}^2 - \kappa^2, \tag{58}$$

and hence from (39) with (57) substituted

$$\epsilon F = (\tilde{\kappa}^2 - \kappa^2) \int_0^a \sin^2 \tilde{\kappa} y \, dy = \tfrac{1}{2}(\tilde{\kappa}^2 - \kappa^2) a. \tag{59}$$

Next, the substitution of (57) into (38) gives

$$\begin{aligned} E &= \frac{1}{2}\int_0^a \frac{\sin^2 \tilde{\kappa} y}{y} dy = \frac{1}{4}\int_0^{2\pi} \frac{1-\cos u}{u} du \\ &= \tfrac{1}{4}\{\gamma + \log 2\pi - \mathrm{Ci}\,(2\pi)\} = 0{\cdot}6094. \end{aligned} \tag{60}$$

(Here γ denotes Euler's constant and Ci the cosine integral.) Finally, the substitution of (54) and (57) into (40) gives

$$\begin{aligned} G &= \tfrac{2}{3}\kappa^2 \int_0^a \frac{\sin^3 \tilde{\kappa} y}{y} dy = \tfrac{2}{3}\kappa^2 \int_0^a \frac{\sin^3 v}{v} dv \\ &= \tfrac{1}{6}\kappa^2\{3\,\mathrm{Si}\,(\pi) - \mathrm{Si}\,(3\pi)\} = 0{\cdot}6468\kappa^2. \end{aligned} \tag{61}$$

(Here Si denotes the sine integral.)

From (34) and (41), using these expressions for E, ϵF and G, we obtain for the stream-function perturbation

$$\begin{aligned} \psi - \Psi_A &= \epsilon\hat{\psi} \\ &= 0{\cdot}773\left(\frac{\tilde{\kappa}^2}{\kappa^2} - 1\right) a \sin\left(\frac{\pi y}{a}\right) \operatorname{sech}^2\{0{\cdot}453(\tilde{\kappa}^2 - \kappa^2)^{\frac{1}{2}} a^{\frac{1}{2}} x\}. \end{aligned} \tag{62}$$

To a first approximation for small amplitude, the radial displacement of the streamlines from their original positions is given by

$$\delta y = -(\psi - \Psi_A)/W = -(\psi - \Psi_A)$$

or

$$\delta r = \frac{1}{r}\,\delta y = -\frac{1}{r}(\psi - \Psi_A). \tag{63}$$

Hence δr is seen to be proportional to

$$\frac{R}{r} \sin\left(\frac{\pi r^2}{R^2}\right),$$

which has a maximum value 1·509 for $r = 0{\cdot}609R$. Now let ϵR denote the magnitude of the maximum displacement δr which is attained at this radius in the central plane $x = 0$ of the wave; thus ϵ is given a precise physical significance consistent with its previous general use as an ordering parameter. Then (62) and (63) show that

$$\epsilon = 0{\cdot}583[(\bar{\kappa}^2/\kappa^2)-1]. \tag{64}$$

Hence we obtain for the displacement

$$\frac{\delta r}{R} = -0{\cdot}663\frac{R}{r}\sin\left(\frac{\pi r^2}{R^2}\right)\operatorname{sech}^2\left\{2{\cdot}635\left(\frac{\epsilon}{1+1{\cdot}71\epsilon}\right)^{\frac{1}{2}}\frac{x}{R}\right\}. \tag{65}$$

Note that this displacement is everywhere inwards towards the axis.

6. Conclusion

It is hoped that this analysis, and particularly the physical discussion included in §3, will reinforce the original presentation of the vortex-breakdown theory, making the theory more acceptable as a basic rationale for the observed phenomenon. Unfortunately, advocacy of this explanation for vortex breakdown is handicapped by the complicated incidental effects that often appear experimentally, notably the very rapid disintegration of the predicted wave trains which has the result that, even in experiments as delicate as Harvey's (1962), only one or two waves are distinguishable. Again, it has often been observed that a filament of dyed fluid originally along the axis goes into a spiralling motion after a breakdown, indicating that the flow structure is neither axisymmetric nor steady. There is good reason to suppose, however, that such effects arise from instability of the steady wave trains described by the present theory and so are secondary. The most reasonable interpretation seems to be that the theory describes the basic state upon which the actual flow subsists; and although sometimes the secondary developments completely obscure this underlying structure of the flow, there are some observations, such as Harvey's, that reveal it unmistakably.

I am indebted to Mr L. E. Fraenkel for many constructive comments on the first draft of this paper.

Appendix

Here it is shown that the results given by method 2, which is based on an evaluation of the flow-force integral to third order, are in accord with a direct solution of the partial differential equations of motion to second order. As before, we assume that the state of flow A is close to critical, so that the expression

$$-H''(\Psi_A)+\frac{I''(\Psi_A)}{2y} = P(y) = \tilde{P}(y)-\epsilon\zeta(y) \tag{A 1}$$

is possible in which ζ has the same order of magnitude as P and $\tilde{P}$. And again writing

$$X = \epsilon^{\frac{1}{2}}x, \tag{A 2}$$

we assume that X-derivatives have the same order of magnitude as the functions differentiated. If the formal expansion

$$\psi = \Psi_A(y) + \epsilon\hat{\psi}_1(X, y) + \epsilon^2\hat{\psi}_2(X, y) + \ldots \tag{A 3}$$

is now substituted into equation (1) and its boundary conditions (2), the coefficients of successive powers of ϵ may be separated and considered in turn.

First we obtain, collecting terms in ϵ^1,

$$\frac{\partial^2\hat{\psi}_1}{\partial y^2} + \tilde{P}(y)\hat{\psi}_1 = 0, \tag{A 4}$$

from which and from the boundary conditions we deduce that

$$\hat{\psi}_1 = g(X)\tilde{\phi}(y), \tag{A 5}$$

where $\tilde{\phi}$ is defined by (12) and (13), and $g(X)$ is an arbitrary function.

Next, the terms in ϵ^2 derived from (1) give

$$\begin{aligned}\frac{\partial^2\hat{\psi}_2}{\partial y^2} + \tilde{P}(y)\hat{\psi}_2 &= -\frac{1}{2y}\frac{\partial^2\hat{\psi}_1}{\partial X^2} + \zeta\hat{\psi}_1 - \tfrac{1}{2}K'''(\Psi_A, y)\hat{\psi}_1^2 \\ &= \left\{-\frac{1}{2y}g_{XX} + \zeta g\right\}\tilde{\phi} - \tfrac{1}{2}K'''(\Psi_A, y)g^2\tilde{\phi}^2.\end{aligned} \tag{A 6}$$

The complementary function for (A 6) satisfying the boundary conditions is simply a multiple of $\hat{\psi}_1$, and so is not required. If we denote the right-hand side of (A 6) by $\Omega(X, y)$, the solution that vanishes at $y = 0$ may be expressed in the form

$$\hat{\psi}_2 = \tilde{\phi}(y)\int_0^y \frac{1}{\tilde{\phi}^2(\bar{y})}\left\{\int_0^{\bar{y}} \Omega(X, \bar{\bar{y}})\tilde{\phi}(\bar{\bar{y}})\,d\bar{\bar{y}}\right\}d\bar{y}. \tag{A 7}$$

The other boundary condition requires $\hat{\psi}_2$ to vanish at $y = a$, where $\tilde{\phi} = 0$ but $d\tilde{\phi}/dy \neq 0$. Hence it is necessary that

$$\int_0^a \Omega(X, y)\tilde{\phi}(y)\,dy = 0. \tag{A 8}$$

(Alternatively, this result follows at once from (A 6) by the well-known theorem used to derive (17) from (16).) Equation (A 8) becomes upon rearrangement

$$Eg_{XX} = Fg - \tfrac{3}{2}Gg^2, \tag{A 9}$$

where E, F and G are the coefficients defined by (38)–(40).

Multiplied by $2g_X$, equation (A 9) is seen to be precisely the first derivative of equation (37). Thus the previous result is confirmed by the present, quite different method of derivation. However, the present approach has the grave disadvantage that in the step from (A 9) to (37) the constant of integration is arbitrary, whereas the previous method very helpfully identified this constant in terms of the flow-force and total-head losses that might occur between the primary and perturbed flows.

Since we could now go on to substitute the expression (A 7) into (A 3) and so obtain an approximation for ψ explicitly to $O(\epsilon^2)$, in this respect the present method gains over the previous one. But the use already made of the second stage of approximation is the more important achievement, since it suffices to fix the essential character of the solution for waves of finite amplitude and permanent form (e.g. it reveals the dependency of wave properties on the supercritical or subcritical condition of the flow, as represented in the coefficient F). Whereas the qualitative results thereby obtained are crucial, nothing of further qualitative significance is gained by adding the correction $\epsilon^2\hat{\psi}_2$ to the particularized first-order perturbation $\epsilon\hat{\psi}_1$.

Inasmuch as a second stage of approximation is needed to determine a functional form that is seemingly arbitrary at a first stage, this is essentially the same kind of perturbation problem that is presented by classical solitary-wave theory. We recall that for the elevation η of a travelling wave on water of depth h, the linearized shallow-water approximation gives

$$\eta = \epsilon f(x - c_0 t), \quad \text{with} \quad c_0 = (gh)^{\frac{1}{2}}.$$

Here f is an arbitrary smooth function. But solitary-wave theory gives

$$\frac{\eta}{h} = \epsilon \operatorname{sech}^2\left\{\left(\frac{3\epsilon}{1+\epsilon}\right)^{\frac{1}{2}} \frac{x - c_1 t}{2h}\right\},$$

with

$$c_1^2 = (1+\epsilon)c_0^2 \quad (\epsilon > 0).$$

(Lamb 1932, §252). This should be compared with (65). The excess $O(\epsilon)$ of the solitary-wave speed c_1 above the critical speed c_0 corresponds to the departure from critical conditions that is represented in the present problem by $\epsilon\zeta = \tilde{P} - P$ and hence by the coefficient F in (A 9) and (37).

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